Adaptive error parameterisation for square root filters

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Introduction

The filtering problem

The filtering problem is solved by computing sequentially (from $p_0(\mathbf{x}_0)$):

$$p_1^f(\mathbf{x}_1), p_1^a(\mathbf{x}_1), \ldots, p_k^f(\mathbf{x}_k), p_k^a(\mathbf{x}_k), \ldots$$

using marginalisation and Bayes' rules:

$$p_k^f(\mathbf{x}_k) = \int p_{k-1}^a(\mathbf{x}_{k-1}) p(\mathbf{x}_k|\mathbf{x}_{k-1}) d\mathbf{x}_{k-1}$$

and

$$p_k^a(\mathbf{x}_k) \propto p_k^f(\mathbf{x}_k) p(\mathbf{y}_k|\mathbf{x}_k)$$

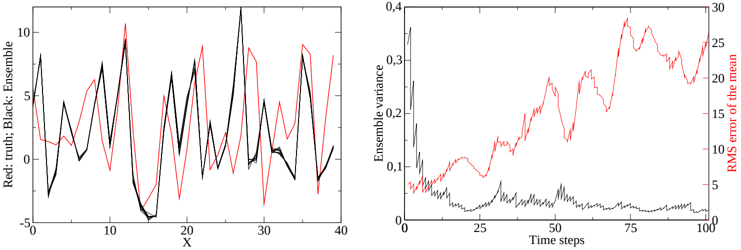
The Kalman filter

Under Kalman's hypothesis, $p_k^f(\mathbf{x}_k) = \mathcal{N}(\mathbf{x}_k^f, \mathbf{P}_k^f)$ and $p_k^a(\mathbf{x}_k) = \mathcal{N}(\mathbf{x}_k^a, \mathbf{P}_k^a)$ with

$$\mathbf{d}_k = \mathbf{y}_k - \mathbf{H}_k \mathbf{x}_k^f \quad (1a)$$
$$\mathbf{C}_k = \mathbf{H}_k \mathbf{P}_k^f \mathbf{H}_k^T + \mathbf{R}_k \quad (1b)$$
$$\mathbf{K}_k = \mathbf{P}_k^f \mathbf{H}_k^T \mathbf{C}_k^{-1} \quad (1c)$$
$$\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{K}_k \mathbf{d}_k \quad (1d)$$
$$\mathbf{P}_k^a = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^f \quad (1e)$$

Introduction

The filter divergence

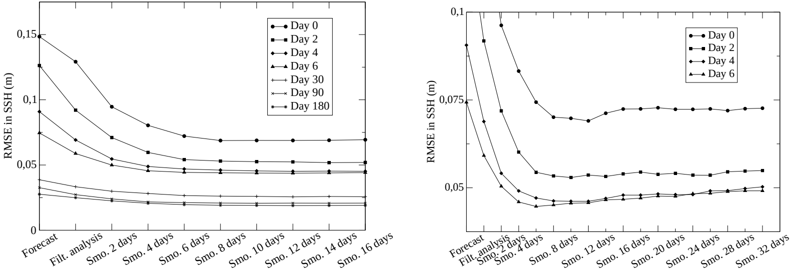


A ill-designed Lorenz-96/EnKF experiment.
Left: truth (red) and ensemble members (black).
Right: RMS error on the mean and total variance.

Reasons: poor initialisation, erroneous rank reduction, inappropriate parameterisation of the model error (if any) or of the observation error...

Introduction

Model error effect on the smoother



With (left) and without (right) parameterisation
of the model error in the sequential smoother.

(Cosme et al, 2010)

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- Perturbations in model parameters and forcings (Evensen, 2003)

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Adaptive methods can help (Dee, 1995).

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Adaptive methods can help (Dee, 1995).
Our approach (Brankart et al, 2009; 2010) follows and extends Dee's (1995). It is tested in an extended "covariance inflation" framework.

Outline of the talk

- Adaptivity: theoretical formulation (9 slides)
- Application 1: Scaling of the forecast covariance matrix (2 slides)
- Application 2: Adjustement of the observation covariance matrix (5 slides)
- Conclusions and perspectives (3 slides)

Adaptivity: theoretical formulation (1)

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- The joint pdfs of α_k and β_k are computed sequentially: $p_0(\alpha_0, \beta_0)$, $p_1^f(\alpha_1, \beta_1)$, $p_1^a(\alpha_1, \beta_1)$, etc, solving a parallel filtering problem;

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- Forecast:

$$p_k^f(\alpha_k, \beta_k) = \int p_{k-1}^a(\alpha_{k-1}, \beta_{k-1}) p(\alpha_k, \beta_k | \alpha_{k-1}, \beta_{k-1}) d\alpha_{k-1} d\beta_{k-1},$$

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- Analysis:

$$p_k^a(\alpha_k, \beta_k) \propto p_k^f(\alpha_k, \beta_k) p(\mathbf{d}_k | \alpha_k, \beta_k),$$

where

$$p(\mathbf{d}_k | \alpha_k, \beta_k) = \mathcal{N}(0, \mathbf{C}_k(\alpha_k, \beta_k)), \mathbf{C}_k(\alpha_k, \beta_k) = \mathbf{H}_k \mathbf{P}_k^f(\alpha_k) \mathbf{H}_k^T + \mathbf{R}_k(\beta_k).$$

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- The pdf of α_k and β_k is updated **before** the pdf of $\mathbf{x}_k$, so that:

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- **Approximation:** $p_k^a(\alpha_k, \beta_k)$ being non Gaussian, Eq. 2 is approximated by:

$$p_k^f(\mathbf{x}_k) \simeq p_k^f(\mathbf{x}_k|\alpha_k = \alpha_k^*),$$

where α_k^* is the best estimate of α_k;

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$$p_k^f(\mathbf{x}_k) \simeq p_k^f(\mathbf{x}_k|\alpha_k = \alpha_k^*),$$

  where α_k^* is the best estimate of α_k;

- The observation pdf is similarly approximated by:

$$p_k(\mathbf{y}_k|\mathbf{x}_k) \simeq p_k(\mathbf{y}_k|\mathbf{x}_k, \beta_k = \beta_k^*).$$

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- We take the MAP (joint) estimates computed by minimizing $J_k(\alpha, \beta) = -\ln[p_k^a(\alpha, \beta)]$;

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Question: How to prescribe the model $p(\alpha_k, \beta_k | \alpha_{k-1}, \beta_{k-1})$?

- Well... Let us try something simple.

Adaptivity: theoretical formulation (4)

A simple model: (perfect) identity

If $p(\alpha_k, \beta_k | \alpha_{k-1}, \beta_{k-1}) = \delta(\alpha_k - \alpha_{k-1}, \beta_k - \beta_{k-1})$, then

$$p_k^a(\alpha, \beta) \propto p_{k-1}^a(\alpha, \beta) p(\mathbf{d}_k | \alpha, \beta) \propto p_0(\alpha, \beta) \prod_{k'=1}^{k} p(\mathbf{d}_{k'} | \alpha, \beta),$$

and

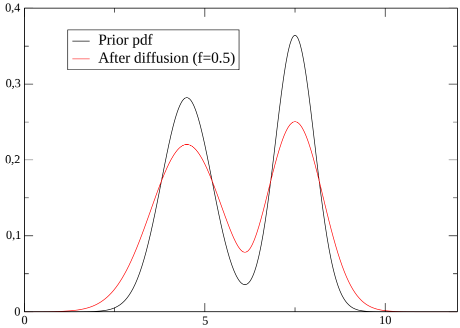
$$\begin{aligned} J_k(\alpha, \beta) &= -\ln[p_0(\alpha, \beta)] && (3) \\ &+ \frac{1}{2} \sum_{k'=1}^{k} \left[\mathbf{d}_{k'}^T \mathbf{C}_{k'}^{-1}(\alpha_k, \beta_k) \mathbf{d}_{k'} + \ln |\mathbf{C}_{k'}(\alpha_k, \beta_k)| \right] && (4) \end{aligned}$$

Adaptivity: theoretical formulation (5)

A slightly less simple model: diffusion model

Consider a "forgetting exponent" $f \leq 1$:

$$p_k^f(\alpha_k, \beta_k) \propto \left[p_{k-1}^a(\alpha_{k-1} = \alpha_k, \beta_{k-1} = \beta_k) \right]^f$$



Adaptivity: theoretical formulation (6)

A slightly less simple model: diffusion model

$$J_k(\alpha_k, \beta_k) = -\ln[p_0(\alpha_k, \beta_k)] \tag{5}$$

$$+\frac{1}{2} \sum_{k'=1}^{k} f^{k-k'} \left[\mathbf{d}_{k'}^T \mathbf{C}_{k'}^{-1}(\alpha_k, \beta_k) \mathbf{d}_{k'} + \ln |\mathbf{C}_{k'}(\alpha_k, \beta_k)| \right] \tag{6}$$

With $f = 0$ and without the first term: solution of Dee (1995).

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With $f = 0$ and without the first term: solution of Dee (1995).

Now: how to deal with this cost function?

Adaptivity: theoretical formulation (7)

Evaluation of the cost function

We draw benefit of the "deterministic" square root or ensemble filter analysis formulation of the SEEK or ETKF type:

- $\mathbf{P}_k^f = \mathbf{S}_k^f {\mathbf{S}_k^f}^T$;
- The rank of $\mathbf{S}_k^f$ is low;
- the inversion of the innovation error covariance matrix using the Sherman-Morrison-Woodbury formula:

$$\left(\mathbf{R} + \mathbf{U}\mathbf{U}^T\right)^{-1} = \mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{U}\left(\mathbf{I} + \mathbf{U}^T\mathbf{U}\right)^{-1}\mathbf{U}^T$$

(Pham et al, 1998; Bishop et al, 2001)

Adaptivity: theoretical formulation (8)

Evaluation of the cost function

This is implemented in practice by introducing:

$$\begin{aligned} \delta_k &= (\mathbf{H}_k \mathbf{S}_k^f)^T \mathbf{R}_k^{-1} \mathbf{d}_k, & (7) \\ \mathbf{U}_k \Lambda_k^{-1} \mathbf{U}_k^T &= (\mathbf{H}_k \mathbf{S}_k^f)^T \mathbf{R}_k^{-1} (\mathbf{H}_k \mathbf{S}_k^f). & (8) \end{aligned}$$

Λ_k^{-1} is diagonal.
Interesting formulation if $\mathrm{rank}((\mathbf{H}_k \mathbf{S}_k^f)^T \mathbf{R}_k^{-1} (\mathbf{H}_k \mathbf{S}_k^f))$ is low and $\mathbf{R}_k$ is easily invertible.

(Pham et al, 1998; Bishop et al, 2001)

Adaptivity: theoretical formulation (9)

Evaluation of the cost function

The cost function can be written:

$$J_k(\alpha_k, \beta_k) = -\ln[p_0(\alpha_k, \beta_k)] + \frac{1}{2} \sum_{k'=1}^{k} f^{k-k'} \left[J^A_{k,k'} + J^B_{k,k'} \right]$$

with

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$$J^A_{k,k'} = \mathbf{d}^T_{k'} \mathbf{R}^{-1}_{k'} \mathbf{d}_{k'} - \delta^T_{k'} \mathbf{U}_{k'} [\mathbf{I} + \Lambda^{-1}_{k'}]^{-1} \mathbf{U}^T_{k'} \delta_{k'}, \qquad (9)$$

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and

$$J^B_{k,k'} = \ln|\mathbf{R}_{k'}| + \mathrm{tr}\{\ln[\mathbf{I} + \Lambda^{-1}_{k'}]\}. \qquad (10)$$

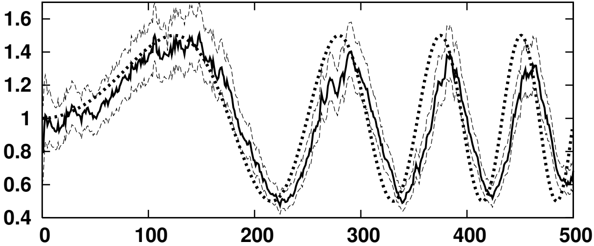
Application 1: Scaling of the forecast covariance matrix (1)

Experiment in a simplified context (NEMO)

- $\mathbf{P}^f$ is formed sampling the time trajectory of a NEMO run;
- A reference trajectory α_k is given;
- A sequence of states is drawn from $\mathcal{N}(\mathbf{x}_{\text{mean}}, \alpha_k \mathbf{P}^f)$;
- SLA fields are perturbed ($\mathbf{R} = \sigma \mathbf{I}$) to form observations;
- The sequence of α_k is retrieved from these innovations.

Application 1: Scaling of the forecast covariance matrix (2)

Experiment in a simplified context (NEMO)



True α_k (dotted line) and retrieval: mode (solid line) and percentiles 0.1 and 0.9 (thin dashed lines).

Application 2: Adjustement of the obs. cov. matrix (1)

Background: Dealing with correlated observation errors

The analysis equations form requires that $\mathbf{R}$ be easily invertible.

(Brankart et al, 2009)

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1. Augment the observation vector with spatial gradients of the observations:

$$y^+ = Ty = \begin{bmatrix} I \\ T_1 \end{bmatrix} y$$

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2. Form a diagonal covariance matrix:

$$R^+ = \begin{bmatrix} \sigma_0^2 I & 0 \\ 0 & \sigma_1^2 I \end{bmatrix}$$

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Assimilating y^+ with R^+ is equivalent to assimilating y with errors correlated according to

$$\mathcal{R}(\rho) = \frac{\sigma_0^2}{2} \exp\left(-\frac{|\rho|}{\ell}\right) \quad \text{with} \quad \ell = \frac{\sigma_0}{\sigma_1}$$

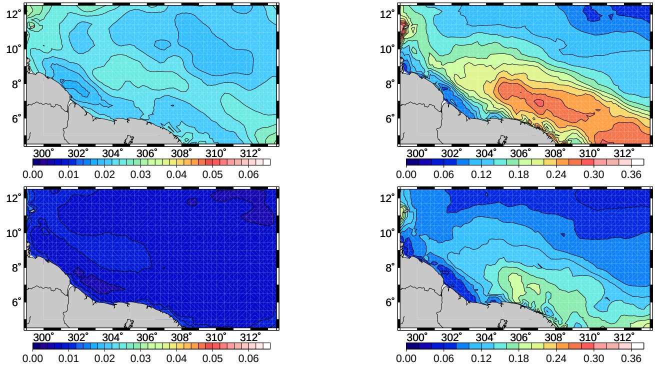
(Brankart et al, 2009)

Application 2: Adjustement of the obs. cov. matrix (2)

Background: Dealing with correlated observation errors

With correlated observation errors, but R is diagonal...

SLA | Current intensity

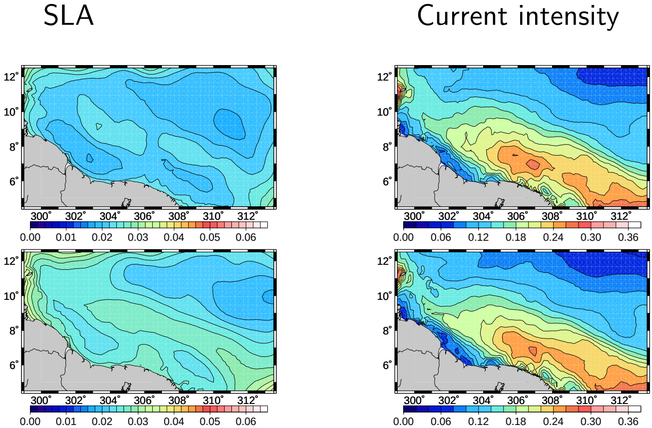


Standard deviation of analysis error: true (top) and estimated by the filter (bottom).

Application 2: Adjustement of the obs. cov. matrix (3)

Background: Dealing with correlated observation errors

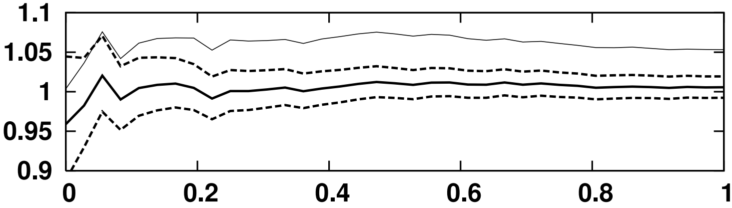
With correlated observation errors, but R is correctly parameterised:



Standard deviation of analysis error: true (top) and estimated by the filter (bottom).

Application 2: Adjustement of the obs. cov. matrix (4)

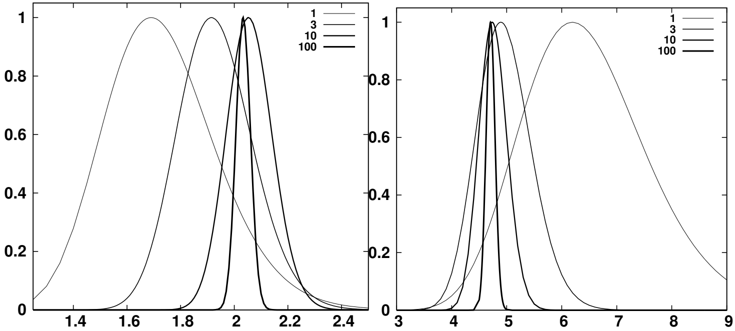
Scaling factor



Estimated scaling factor for the observation error covariance matrix (mode and percentiles 0.1 and 0.9; truth=1).

Application 2: Adjustement of the obs. cov. matrix (5)

Correlation length scale



Likelihood function for the correlation length scale as a function of the number of input innovation vectors (1, 3, 10, 100). l =2 and 5 (FOAR correlation model).

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Summary

- A general adaptive scheme has been developed and implemented with the SEEK filter.
- The numerical cost is low due to the SEEK formulation.
- It can be used in conjunction with other model error parameterisations.
- Tests in an extended "covariance inflation" case have been successful.

Conclusions and perspectives (2)

To do list

The cost function

$$J_k(\alpha) = \mathbf{d}_k^T \mathbf{C}_k^{-1}(\alpha_k)\mathbf{d}_k + \ln|\mathbf{C}_k(\alpha_k)|$$

is minimum for

$$\mathbf{C}_k(\alpha_k^*) = \mathbf{d}_k \mathbf{d}_k^T$$

or

$$\alpha_k^* \mathbf{H}_k \mathbf{P}_k^f \mathbf{H}_k^T = \mathbf{d}_k \mathbf{d}_k^T - \mathbf{R}_k$$

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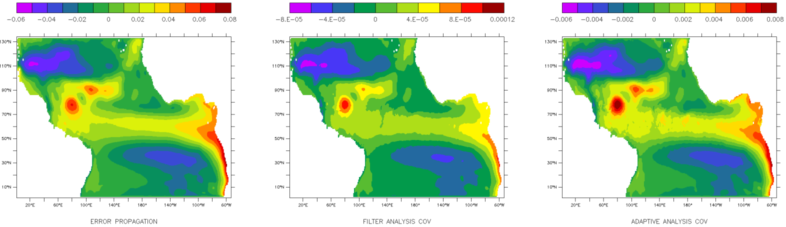
- Find how to avoid negative α_k (a clean way).
- Formulate for a localised analysis scheme with observational gaps.
- Finalise implementation with a real ocean model and test the effect with the smoother.

Conclusions and perspectives (3)

First experiment with a realistic ocean model (NEMO)



One mode of error: Background (left), analysed (centre), adaptively inflated (right).