Extreme Value Theory in a Bayesian framework

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- E2C2: Pascal Yiou, Pierre Ribereau, Marta Nogaj, etc

- Doug Nychka NCAR
Time Series Analysis for Maxima

Convergence of sample maxima

Normal density $\Rightarrow$ Gumbel density

Uniform density $\Rightarrow$ Weibull density

Cauchy density $\Rightarrow$ Fréchet density

$n = 50$  
$n = 100$
Bayesian Statistics

Differences with frequentist methods:
- Parameters are treated as random variables
- Incorporation of prior beliefs about the parameters
- More flexible but more computer intensive – MCMC methods
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\[
P(B|A) = P(A|B) \times \frac{P(B)}{P(A)}
\]

\[
P(\theta|Y) = \frac{P(Y|\theta) \times P(\theta)}{P(Y)} \propto P(Y|\theta) \times P(\theta)
\]

with \( \theta = (\xi, \Phi, \ldots) \) and \( Y = \text{Precip} \)
Goal:
To produce a map of extreme precipitation return levels for Colorado’s Front Range. Flood planners currently use the 1973 Precipitation Atlas.

Definition of the return level:
The level is expected to be exceeded on average once every $T$ years.

Methodology:
We model large precipitation via Extreme Value Theory and take into account of the spatial variability with a Bayesian-based approach.
Bayesian Hierarchical Model

Model Assumptions:
A) Large precipitation events $Y_j(x_i)$ follow a GPD

$$P_{\theta}\{Y_j(x_i) - u > y|Y_j(x_i) > u\} = \left(1 + \frac{\xi_i y}{\exp \phi_i}\right)^{-1/\xi_i}$$

B) Bayesian Structure:
The GPD parameters $\theta_i = (\phi_i, \xi_i)$ are considered as random variables:
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The GPD parameters $\theta_i = (\phi_i, \xi_i)$ are considered as random variables:

$$
\phi_i = \alpha_0 + \alpha_1 \times \text{elevation}_i + \text{MVN} \left(0, \beta_0 \exp(-\beta_1 ||x_k - x_j||)\right)
$$
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$$

$$
\xi_i = \begin{cases} 
\xi_{\text{mountains}}, & \text{if } x_i \in \text{mountains} \\
\xi_{\text{plains}}, & \text{if } x_i \in \text{plains}
\end{cases}
$$
Bayesian Hierarchical Model

\[ \alpha_0 + \alpha_1 z \]
\[ \beta_0 \exp(-\beta_1 \| \cdot \|) \]
\[ P(Y(x) > u) \]
\[ \xi_{\text{moutains}} \]
\[ \xi_{\text{plains}} \]

\text{Priors} \rightarrow \begin{array}{c}
\phi \\
\xi \\
\end{array} \rightarrow \begin{array}{c}
r(x) \\
P(Y(x) > u) \\
\xi_{\text{moutains}} \\
\xi_{\text{plains}} \\
\end{array} \leftarrow \text{Priors}
Bayesian Hierarchical Model

Priors

- \((\alpha_0, \alpha_1)\) Non-informative priors (Uniform\((-\infty, \infty))\)
- \((\beta_0, \beta_1)\) Informative priors (Uniform\((a, b)\) with \(0 < a < b < 1\))
- \((\xi_{\text{moutains}}, \xi_{\text{plains}})\) Gaussian distribution with non-informative mean and variance
Prior (dashed) & posterior (solid)

Boulder Station

phi

density

3.2 3.4 3.6 3.8 4.0
0 2 4 6 8
Posterior Mean of 25y-return levels
Quantile Posterior return levels

![Map of Quantile Posterior return levels with locations of Ft. Collins, Boulder, Denver, Colo Spgs, and Pueblo marked. The color scale ranges from 0.5 to 1.2.]
## Model Selections

<table>
<thead>
<tr>
<th>Models in Latitude/Longitude Space</th>
<th>$\bar{D}$</th>
<th>$p_D$</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scale $=\phi$ Shape $=\xi$</td>
<td>112264.2</td>
<td>2.0</td>
<td>112266.2</td>
</tr>
<tr>
<td>Scale $=\alpha_0 + \alpha_1(elev)$ Shape $=\xi$</td>
<td>98528.8</td>
<td>30.4</td>
<td>98559.2</td>
</tr>
<tr>
<td>Scale $=\alpha_0 + \alpha_1(elev) + \alpha_2(msp)$ Shape $=\xi$</td>
<td>98529.7</td>
<td>29.6</td>
<td>98559.6</td>
</tr>
<tr>
<td>Scale $=\alpha_0 + \alpha_1(elev)$ Shape $=ξ_{mtn}, ξ_{plains}$</td>
<td>98526.4</td>
<td>31.8</td>
<td>98558.2</td>
</tr>
<tr>
<td>Scale $=\begin{cases} ξ_{mtn} + \alpha_{m1}(elev) \ ξ_{plains} + \alpha_{p1}(elev) \end{cases}$</td>
<td>98524.0</td>
<td>25.8</td>
<td>98549.8</td>
</tr>
</tbody>
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<table>
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<tr>
<td>Scale $=\alpha_0 + \alpha_1(elev)$ Shape $=ξ_{mtn}, ξ_{plains}$</td>
<td>98524.0</td>
<td>26.0</td>
<td>98550.0</td>
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<td>Scale $=\alpha_0 + \alpha_1(elev) + \alpha_2(msp)$ Shape $=ξ_{mtn}, ξ_{plains}$</td>
<td>98523.6</td>
<td>27.3</td>
<td>98550.9</td>
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Spatial Statistics for Extremes

A few Approaches for modeling spatial extremes

- **Max-stable processes**: Adapting asymptotic results for multivariate extremes
  Schlather & Tawn (2003), Naveau et al. (2005), de Haan & Pereira (2005)

- **Bayesian or latent models**: spatial structure indirectly modeled via the EVT parameters distribution
  Coles & Tawn (1996), Cooley et al. (2005)

- **Linear filtering**: Auto-Regressive spatio-temporal heavy tailed processes,
  Davis and Mikosch (2005)

- **Gaussian anamorphosis**: Transforming the field into a Gaussian one
  Wackernagel (2003)
Take-home messages

- Improvement over the 1973 Colorado return level atlas
- Uncertainty represented by posterior distributions
- The hierarchical Bayesian framework provides a rich and flexible family for modeling extreme behavior
- Very much data-oriented approach

Future research

- Develop spatio-temporal for other regions
- Derive statistical schemes for downscaling of extremes

Acknowledgement

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E2C2: Extreme Events, Cause and Consequences
Precipitation in Colorado

Data:

- 56 weather stations in Colorado (semi-arid and mountainous region)
- Daily precipitation for the months April-October
- Time span = 1948-2001
- Precision: 1971 from 1/10th of an inch to 1/100
- Not all stations have the same number of data points
The Independent Approach

\[ Y_j(x_i) = \text{Precipitation amount recorded at the location at } x_i \text{ on day } j \]

**Model Assumptions:**

Large precipitation events \( Y_j(x_i) \) which exceed a threshold \( u \) follow a

**Generalized Pareto Distribution (GPD)**

\[
P\{Y_j(x_i) - u > y | Y_j(x_i) > u\} = \left(1 + \frac{\xi_i y}{\exp \phi_i}\right)^{-1/\xi_i}
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Bayesian Model with one level

\[ P(\theta | Y) \propto P(Y | \theta) \times P(\theta) \]

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Hierarchical Bayesian Model with two levels

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Parameters Estimation

- MCMC methods (Gibbs samplers, etc) Robert & Casella, 1999