On spatial extremes: with application to a rainfall problem

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Abstract

We consider daily rainfall observations at 32 stations in the province of North Holland (The Netherlands) during 30 years. Let Q be the *total* rainfall in this area on one day. An important question is: what is the amount of rainfall Q that is exceeded once in 100 years? This is clearly a problem belonging to extreme value theory. Also it is a genuinely spatial problem.

Recently, a theory of extremes of continuous stochastic processes has been developed. Using the ideas of that theory and much computer power (simulations) we have been able to come up with a reasonable answer to the question above.

Keywords: spatial extremes, max-stable process, areal reduction factor.

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1 Introduction

Extreme rainfall statistics are frequently used when a damaging flood has occurred to answer questions about the rarity of the event. Engineers often need extreme rainfall statistics for the design of structures for flood protection. A typical question is e.g. what is the amount of rain in a given area on one day that is exceeded once in 100 years? Or, more mathematically, what is the 100-year quantile of the total rainfall in the area on one day? In this paper this question is investigated for a low-lying flat area in the northwest of the Netherlands. The area is shown in Figure 1. Because it roughly covers the province of North Holland, it will shortly be indicated as North Holland.

There are 32 rainfall stations in the area for which daily data were available for the 30-year period 1971-2000. Only the fall season, i.e. the months September, October and November, is considered. In this season the likelihood of flooding and its impact are relatively large. Because of the restriction to the fall season it is reasonable to assume stationary in time. Stationary in space, except for location and scale, is also assumed.

Since we have to extrapolate from a 30-year to a 100-year period, our problem is an extreme value problem - in the absence of detailed and tractable physical models.

Coles and Tawn, in a series of papers (Coles [8], Coles and Tawn [9]) have developed methods to deal with spatial extremes based on the spectral representation (de Haan [11], see also Schlather [26]). Recently, Cooley, Nychka and Naveau [10] presented a Bayesian framework for dependence in spatial extremes.

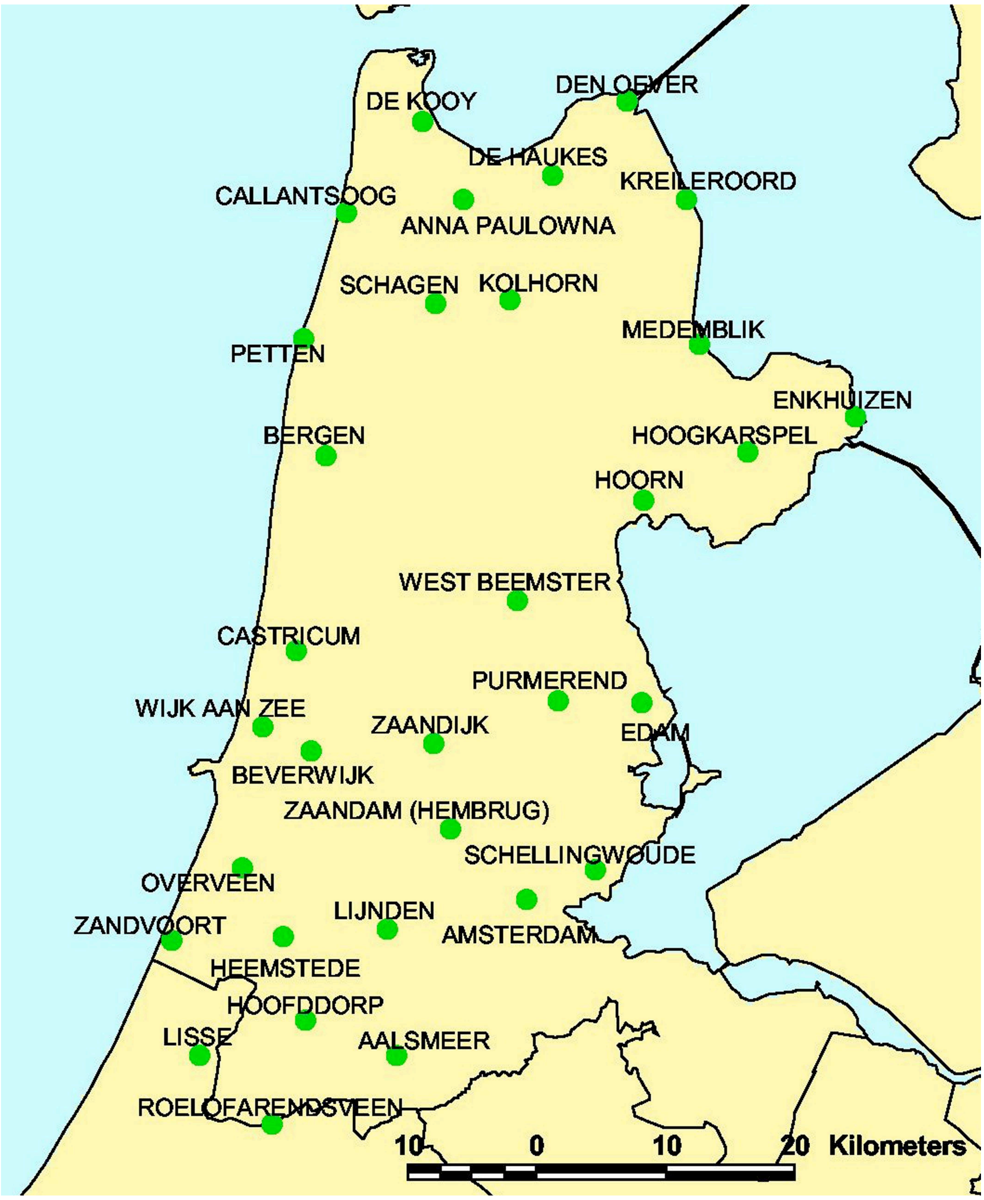

Figure 1: The Study Area: North Holland

Engineers often make use of areal reduction factors (ARFs) to convert quantiles for point rainfall to the corresponding quantiles of areal rainfall. ARFs have been derived empirically by estimating the areal rainfall as a function of point rainfall measurements (e.g., Natural Environment Research Council (NERC) [23]; Bell [4]) or by statistical modelling (e.g., Bacchi and Ranzi [2]; Sivapalan and Blöschl [27]; Veneziano and Langousis [28]). The latter requires assumptions on distributions, spatial correlation and/or scaling behavior.

There exists a completely non-parametric way of estimating the probability of a "failure set" or "extreme set" in the space of continuous functions (de Haan and Ferreira [13], section 10.5). It involves standardizing the marginal distributions and then pulling back the set until the set contains some observations, taking advantage of the approximate homogeneity property of the set as in the finite-dimensional case (cf. e.g. de Haan and de Ronde [12]). However it is rather impractical to try to apply that method here: the failure set is simply too complicated due to the integration of the rainfall process. Instead we shall adopt a reasonable *model* and solve the problem by simulating synthetic daily rainfall fields with the estimated model.

In order to motivate our solution we first explain some relevant aspects of extreme value theory, in $\mathbb{R}^1$, $\mathbb{R}^d$ $(d > 1)$ and $C[0,1]$ (section 2). In section 3, we specify the stochastic process used in the simulation. This process is used only to simulate "extreme" rainfall. For non-extreme rainfall we sample from the available data. In section 4 we explain how we combine the two to get a simulated day of rainfall. The estimation of the dependence parameter is dealt with in section 5. Section 6 discusses the outcome of the simulation and the answer to our problem. Section 7 summarizes our main conclusions. Proofs are given in two Appendices.

2 Extreme Value Background

We now explain the background of our approach by reviewing some aspects of extreme value theory and the related theory of excursions over a high threshold. This will be done first in the one-dimensional case (Section 2.1), then the finite-dimensional case (Section 2.2) and finally the case of continuous stochastic processes (Section 2.3). The results in the various cases are similar but of increasing complexity. That is why we start with the one-dimensional case which is well-known (Gnedenko [20] and Pickands [25] respectively).

2.1 One-dimensional Space

Suppose that the distribution function F is in the domain of attraction of an extreme value distribution, i.e., if $X_1, X_2, \cdots$ are i.i.d. with distribution function F, there are a positive function a and a function b, such that

$$\lim_{n\to\infty} P\left(\max_{1\le i\le n} \frac{X_i - b(n)}{a(n)} \le x\right) = G(x)$$

a non-degenerate distribution function. We denote this by $F \in \mathcal{D}$. Then a and b can be chosen such that

$$G(x) = G_\gamma(x) = \exp\left\{-(1+\gamma x)^{-1/\gamma}\right\}$$

for all x with $1 + \gamma x > 0$. Then we also say $F \in \mathcal{D}(G_\gamma)$.

Let X be a random variable with distribution function F. Then there exists a positive function a and real shape parameter γ (the *extreme value index*), such that for all x with $1+\gamma x > 0$,

$$\lim_{t \uparrow x^*} P\left(\frac{X-t}{a(t)} > x | X > t\right) = (1+\gamma x)^{-1/\gamma} =: 1 - Q_\gamma(x).$$

Here $x^* := \sup\{x : F(x) < 1\}$. This means that the larger observations in a sample follow approximately the probability distribution Q_γ - the generalized Pareto distribution (GPD). (c.f. Balkema and de Haan [3]; Pickands [25]) Note that $1 - Q_\gamma(x) = -\log G_\gamma(x)$.

Let R be a random variable with distribution function Q_γ. Then,

$$P\left(\frac{R-t}{1+\gamma t} > x | R > t\right) = P(R > x)$$

for those x and t for which $1+\gamma t > 0$ and $1+\gamma x > 0$. We can call this property *excursion stability.*

Suppose that we have observed a sample $X_1, X_2, \cdots, X_n$ from F. Since this is a completely specified probability distribution, it is possible to use it as a basis to simulate more "large observations", even larger than those in the sample. Hence, by resampling the non-extreme part of the sample and simulating extreme observations from the GPD distribution one can produce more and more "observations", even extreme ones. This idea of using partly simulation and partly resampling is the main idea behind what we intend to do.

We finish this part with two remarks.

Remark 2.1. *For the simulation, we need estimators for the three main "parameters": the shape parameter γ, the scale pseudo-parameter a and* *the location pseudo-parameter b. See Remark 3.1 below. Various estimators have been proposed and studied in the literature.*

Remark 2.2. *The generalized Pareto distribution Q_γ is not the extreme value distribution, but is related to the extreme value distribution.*

2.2 Finite-dimensional Space

Let us now consider the finite-dimensional case, or rather the two-dimensional case for simplicity. Let (X, Y) be a random vector with distribution function F. Suppose $F \in \mathcal{D}$, i.e. if $(X_1, Y_1), (X_2, Y_2), \cdots$ are i.i.d. with distribution function F, there are functions b and d and positive functions a and c, such that

$$\lim_{n\to\infty} P\left(\max_{1\le i\le n} \frac{X_i - b(n)}{a(n)} \le x, \max_{1\le i\le n} \frac{Y_i - d(n)}{c(n)} \le y\right) = G(x, y),$$

a distribution function with non-degenerate marginals. If this is the case, we say $F \in \mathcal{D}(G)$ and G is a (multivariate) extreme value distribution. Then, as in the one-dimensional case, there exists a related 2-dimensional GPD distribution Q_H, obtained for example as follows:

$$\begin{aligned}&\lim_{t\to\infty} P\left(\frac{X - b(t)}{a(t)} > \frac{x^{\gamma_1} - 1}{\gamma_1} \quad \text{or} \quad \frac{Y - d(t)}{c(t)} > \frac{y^{\gamma_2} - 1}{\gamma_2} | X > b(t) \quad \text{or} \quad Y > d(t)\right)\\ =&2\int_0^1 \max\left(\frac{s}{x}, \frac{1-s}{y}\right) H(ds) =: 1 - Q_H(x, y),\end{aligned}$$

for $(x, y) \in D_H = \left\{(x, y) : 2\int_0^1 \max\left(\frac{s}{x}, \frac{1-s}{y}\right) H(ds) \le 1\right\} \supset \{(x, y) : x, y \ge 2\}$, where γ_1 and γ_2 are the marginal extreme value indices, and H is a probability distribution function on $[0, 1]$ with mean $1/2$. Any distribution H with mean $1/2$ may occur. (c.f., e.g. de Haan and Ferreira [13], Chapter

6) Similar to the one-dimensional case we have

$$-\log G\left(\frac{x^{\gamma_1}-1}{\gamma_1},\frac{y^{\gamma_2}-1}{\gamma_2}\right)=1-Q_H(x,y).$$

Q_H is a probability distribution function on D_H with the properties:

1. Standard one-dimensional GPD marginals: $Q_H(x,\infty)=Q_H(\infty,x)=1-1/x$, for $x\geq 1$;

2. Homogeneity: $1-Q_H(tx,ty)=t^{-1}(1-Q_H(x,y))$ for $t>1$ and $(x,y)\in D_H$, in particular $Q_H\in\mathcal{D}$:

$$Q_H^n(nx,ny)=(1-(1-Q_H(x,y))/n)^n\rightarrow\exp\{-(1-Q_H(x,y))\}=G\left(\frac{x^{\gamma_1}-1}{\gamma_1},\frac{y^{\gamma_2}-1}{\gamma_2}\right);$$

3. Excursion stability: If (R,Q) is a random vector with distribution function Q_H, then with $c:=1-Q_H(1,1)$, we have for $x,y\in D_H$, $t>c$

$$P\left(R>\frac{tx}{c}\ \text{ or }\ Q>\frac{ty}{c}|R\vee Q>t\right)=P(R>x\ \text{ or }\ Q>y).$$

We remark that a random vector with an extreme value distribution can be constructed as follows. Consider a Poisson point process on $(0,\infty)$ with mean measure $r^{-2}dr$. Let $\{Z_1,Z_2,\cdots\}$ be a realization of this point process. Let V be a random variable with distribution function H and consider i.i.d. copies $V_1,V_2,\cdots$ of V. Then the random vector

$$\left(\bigvee_{i=1}^{\infty}2Z_iV_i,\bigvee_{i=1}^{\infty}2Z_i(1-V_i)\right)$$

has an extreme value distribution and both marginals have distribution function $\exp(-1/x),x>0$.

We want to follow the line of reasoning from the one-dimensional situation and propose to use Q_H to simulate more "large observations", to be combined with resampling from the available sample. However, simulation from a multivariate distribution is more complicated than in the one-dimensional case. It is more convenient if we can find a random vector that is easy to simulate and that has the same distribution. Consider the random vector $(2YV, 2Y(1-V))$ with Y and V independent, Y has distribution function $1-1/x$, $x \geq 1$ and V has distribution function H. It is easy to check that the distribution function $Q^0_H(x,y)$ of $(2YV, 2Y(1-V))$ coincides with $Q_H(x,y)$ for $x, y \geq 2$. The fact that the distribution function is not exactly the same is not a problem: we are dealing with an asymptotic problem and the important thing is that Q^0_H and Q_H have the same asymptotic dependence function, i.e.

$$\lim_{t\to\infty} \frac{1-Q_H(tx,ty)}{1-Q^0_H(tx,ty)} = 1 \tag{1}$$

for $x, y > 0$. In fact any distribution function in the domain of attraction of G would do since the asymptotic dependence structure is the same as for the limiting extreme value distribution.

Now the random vector $(2YV, 2Y(1-V))$ is useful but not flexible enough: the set of conditions $V \in [0,1]$ and $EV = 1/2$ is rather restrictive. Hence let us consider the random vector (YA_1, YA_2) with Y and (A_1, A_2) independent, Y as before and A_1 and A_2 positive with $EA_1 = EA_2 = 1$. The distribution function Q^* of (YA_1, YA_2) satisfies the following properties.

1^*. $1-Q^*(x,\infty) = E\min\left(1, \frac{A_1}{x}\right)$ for $x > 0$, hence $\lim_{t\to\infty} t(1-Q^*(tx,\infty)) = 1/x$, similarly for $Q^*(\infty, x)$;

2^*. $\lim_{t\to\infty} t(1-Q^*(tx,ty)) = E\frac{A_1}{x} \vee \frac{A_2}{y}$ for $x, y > 0$, i.e. $Q^* \in \mathcal{D}$;

3^*.

$$\lim_{t\to\infty} P(YA_1 > tx/c \ \text{ or } \ YA_2 > ty/c | YA_1 > t \ \text{ or } \ YA_2 > t) = E\frac{A_1}{x} \vee \frac{A_2}{y}$$

for $x, y > 0$ with $c := EA_1 \vee A_2$.

Here Q^* is easy to simulate, but it satisfies only approximately (not exactly) the three properties 1, 2 and 3. Because of property 2^* (meaning that the distribution function of (YA_1, YA_2) has the same asymptotic dependence function as the distribution function $\max\left(0, 1 - E\frac{A_1}{x} \vee \frac{A_2}{y}\right)$, c.f. (1)), we can still use Q^* for simulation albeit with caution.

2.3 Extremes of Continuous Stochastic Processes

What do we mean by extremes in $C[0,1]$? The setup is as follows. Let $\{X(s)\}_{s\in[0,1]}$ be a stochastic process in $C[0,1]$. Consider independent copies $X_1, X_2, \cdots$ of the process X. Compose the continuous stochastic processes

$$\left\{\max_{1\le i\le n} X_i(s)\right\}_{s\in[0,1]}.$$

Suppose that for some positive functions $a_s(n)$ and real functions $b_s(n)$, the sequence of processes

$$\left\{\max_{1\le i\le n} \frac{X_i(s) - b_s(n)}{a_s(n)}\right\}_{s\in[0,1]}$$

converges in $C[0,1]$. If this is the case, we say $X \in \mathcal{D}$. Let us call the limiting process $\{U(s)\}_{s\in[0,1]}$. Then we say $X \in \mathcal{D}(U)$. The following proposition is useful for our purposes (de Haan and Lin [14]).

Proposition 2.1. *$X \in \mathcal{D}$ if and only if the following two statements hold:*

1. (uniform convergence of the marginal distributions) There exists a continuous function $\gamma(s)$ such that, for $x > 0$

$$\lim_{n\to\infty} P\left(\max_{1\le i\le n} \frac{X_i(s) - b_s(n)}{a_s(n)} \le \frac{x^{\gamma(s)} - 1}{\gamma(s)}\right) = \exp\left(-\frac{1}{x}\right),$$

uniformly for $s \in [0,1]$.

2. (convergence of the standardized process) With $F_s(x) := P(X(s) \leq x)$ *for* $s \in [0,1]$,

$$\left\{ \max_{1 \leq i \leq n} \frac{1}{n(1 - F_s(X_i(s)))} \right\} \xrightarrow{d} \{\eta(s)\} \quad \textit{(say)}$$

in $C[0,1]$. *Note that all one-dimensional marginal distributions of the process* $1/(1 - F_s(X_i(s)))$ *are equal to* $1 - 1/x$, $x \geq 1$.

Remark 2.3. *The process* η *satisfies: if* $\eta_1, \eta_2, \cdots$ *are i.i.d. copies of* η, *then*

$$\frac{1}{n} \bigvee_{i=1}^{n} \eta_i \stackrel{d}{=} \eta,$$

i.e. the process is simple max-stable. *(The word "simple" indicates that all marginal distributions are standard Fréchet distributions,* $\exp(-1/x), x > 0$.*)*

Remark 2.4.

$$\{U(s)\} \stackrel{d}{=} \left\{ \frac{(\eta(s))^{\gamma(s)} - 1}{\gamma(s)} \right\}.$$

As a consequence of this proposition, we can study the "simple" process η first and go back to U later, in a straightforward way.

Two relatively simple characterizations of simple max-stable processes are known. One of them can serve our purposes. It is given in the following proposition.

Proposition 2.2. *(de Haan and Ferreira [13] Corollary 9.4.5) Every simple max-stable process* η *in* $C[0,1]$ *can be generated in the following way. Consider a Poisson point process on* $(0, \infty]$ *with mean measure* $r^{-2}dr$. *Let* $\{Z_i\}_{i=1}^{\infty}$ *be a realization of this point process. Further consider i.i.d.*

positive stochastic processes $V, V_1, V_2, \cdots$ *in* $C[0,1]$ *with* $EV(s) = 1$ *for all* $s \in [0,1]$ *and* $E \sup_{0 \le s \le 1} V(s) < \infty$. *Let the point process and the sequence* $V, V_1, V_2, \cdots$ *be independent. Then*

$$\eta \stackrel{d}{=} \bigvee_{i=1}^{\infty} Z_i V_i.$$

Conversely, each process with this representation is simple max-stable. One can take the stochastic process V *such that*

$$\sup_{0 \le s \le 1} V(s) = c \quad a.s.$$

with c *some positive non-random constant.*

Now recall the "generalized Pareto" results in one- and finite-dimensional extremes, that allowed us to simulate from the tail of the distribution. What is the situation in this spatial setup?

One way to proceed is as in the finite-dimensional case. Let Y be a random variable with distribution function $1 - 1/x$, $x \ge 1$ (i.e. one-dimensional GPD). Let V be a positive stochastic process in $C[0,1]$ that satisfies the conditions of Proposition 2.2: $EV(s) = 1$ for $s \in [0,1]$ and $\sup_{0 \le s \le 1} V(s) = c$, a non-random constant. Let Y and V be independent. Consider *the GPD process*

$$\{\xi(s)\}_{s \in [0,1]} := \{YV(s)\}_{s \in [0,1]} .$$

The process ξ is in $C[0,1]$ and satisfies

1. Standard GPD tail: $P(YV(s) > x) = 1/x$ for $x > c$;
2. Homogeneity, in particular $\xi \in \mathcal{D}(\eta)$;
3. Excursion stability: The distribution of $\{cYV(s)/t\}$ given $\sup_{0 \le s \le 1} YV(s) > t$ is the same as that of $\{YV(s)\}$ for $t > c$.

The third property depends on the condition that $\sup_{0\leq s\leq 1} V(s) = c$, a non-random constant. If we only know $E\sup_{0\leq s\leq 1} V(s) < \infty$, the property does not apply, but we still have the three properties in an approximate sense as in the finite-dimensional case.

We remark that the stochastic process $\{YV(s)\}$ is in the domain of attraction of the process $\{\eta(s)\}$, hence the asymptotic dependence structure of the two processes is the same (c.f. section 2.2) and either of the processes can be used for simulating extreme events. This is also true for the process $\{YV(s)\}$ with the weaker side condition $E\sup_{0\leq s\leq 1} V(s) < \infty$. Hence there are three candidate processes for simulating extremal rainfall.

We finish this section with a remark

Remark 2.5. *In all of the above we can replace $[0,1]$ by any compact subset of an Euclidean space, i.e. we can deal with spatial extremes.*

3 Stochastic Process for Simulating "Extreme" Rainfall

- The starting point for the simulation of the rainfall process is Proposition 2.2, the representation of simple max-stable processes and its counterpart, the excursion stable process $\{YV(s)\}$. Conceptually, as explained in Section 2, the excursion stable process is the right one to use.

However, non-parametric estimation of the characteristics of the process V is presently beyond our reach. Hence we choose to work with a tractable parametric model for V. Unfortunately, the condition $\sup_{0\leq s\leq 1} V(s) = c$, that makes the process $\{YV(s)\}$ excursion stable, is very stringent and we could not find a reasonable parametric model for such a process. Hence, it seems better to stay with the model $\{YV(s)\}$ but replace the condition $\sup_{0\le s\le 1} V(s) = c$ by $E\sup_{0\le s\le 1} V(s) < \infty$ as allowed by Proposition 2.2. Then the excursion stability is still approximately true, i.e. the process has the same asymptotic dependence structure. But we meet another problem. In order to tie the simulated process to the observed non-extreme rainfall, it is imperative that the marginal distributions of the simulated process has a GPD tail (c.f. relation (6) below). As explained in section 2, this is not correct for $\{YV(s)\}$ with $E\sup_{0\le s\le 1} V(s) < \infty$, worse, the marginal distribution is quite untractable, hence a transformation to repair this problem seems difficult to find.

Only the third possibility remains: to choose the simple max-stable process from Proposition 2.2 for the simulation. Then the asymptotic dependence structure of the process is more or less the same as that of the corresponding GPD-type process $\{YV(s)\}$ and the marginal distributions are all the same hence they can easily be transformed to the distribution function $1 - 1/x$, $x \ge 1$.

- This is what we do in the simulation. For V we choose the so-called exponential martingale (c.f. Øksendal [24], excercise 4.10). Also we have to extend the process to a process with a two-dimensional index set. We choose the model

$$\eta(s_1, s_2) := \bigvee_{i=1}^{\infty} Z_i \exp\{W_{1i}(\beta s_1) + W_{2i}(\beta s_2) - \beta(|s_1| + |s_2|)/2\} \quad (2)$$

for $(s_1, s_2) \in \mathbb{R}^2$ (or rather the area under study, North Holland). Here

$\{Z_i\}$ is the realization of a Poisson point process on $(0, \infty)$ with mean measure $r^{-2}dr$. The processes $W_{11}, W_{21}, W_{12}, W_{22}, W_{13}, W_{23}, \cdots$ are independent copies of double-sided Brownian motions W defined as follows. Take two independent Brownian motions B_1 and B_2. Then

$$W(s) := \begin{cases} B_1(s), & s \geq 0; \\ B_2(-s), & s < 0. \end{cases} \tag{3}$$

The positive constant β reflects the amount of spatial dependence at high levels of rainfall: "β small" means strong dependence and "β large" means weak dependence. The model assumes that the dependence between extreme rainfall at two locations depends only on the distance between the locations as we shall see later on.

The process η satisfies the requirements of Proposition 2.2:

$$E \exp \{W_1(\beta s_1) + W_2(\beta s_2) - \beta(|s_1| + |s_2|)/2\} = 1 \quad \text{for} \quad (s_1, s_2) \in \mathbb{R}^2,$$

and

$$E \sup_{\substack{a_1 \leq s_1 \leq b_1 \\ a_2 \leq s_2 \leq b_2}} \exp \{W_1(\beta s_1) + W_2(\beta s_2) - \beta(|s_1| + |s_2|)/2\} < \infty \quad \text{for all } a_1 < b_1, a_2 < b_2 \text{ real.}$$

By Proposition 2.2, the one-dimensional marginal distributions of (2) are all $e^{-1/x}, x > 0$. In Appendix B we calculate the two-dimensional distributions. They are invariant under a shift. The same holds for the higher-dimensional marginal distributions (the proof is in Appendix A). Hence the process is shift stationary as it should be for our application.

- For the simulation of our process we need to simulate a Poisson point process. Simulating a Poisson point process is laborious. That is why we apply the following simplification.

$\{Z_i\}$ is a Poisson point process on $(0, \infty)$ with mean measure $r^{-2}dr$, hence $\{1/Z_i\}$ is a Poisson point process on $(0, \infty)$ with mean measure dr. This is a homogeneous Poisson point process on $(0, \infty)$ and the points can be constructed as partial sums of exponential random variables (we then get the points in increasing order):

$$E_1, \; E_1 + E_2, \; E_1 + E_2 + E_3, \cdots$$

with $E_1, E_2, \cdots$ i.i.d. standard exponential.

Hence, since the second factors in (2) are i.i.d. stochastic process, we can exhibit (2) as

$$\eta(s_1, s_2) := \bigvee_{i=1}^{\infty} \frac{1}{E_1 + E_2 + \cdots + E_i} \exp\{W_{1i}(\beta s_1) + W_{2i}(\beta s_2) - \beta(|s_1| + |s_2|)/2\} \tag{4}$$

This is somewhat analogous to LePage's representation for stable processes (LePage, Woodroofe and Zinn [22]).

Since the first factors in (4) form a decreasing sequence, one can approximate the process η by taking the maximum of only finitely many points. In fact it turns out that even 4 points are enough to get a reasonable result.

- We have now a simple max-stable process that can be simulated rather well. But - taking into account our discussion of the finite-dimensional case - in fact we need a process that has generalized Pareto marginals, not the standard Fréchet extreme value distribution as in Remark 2.3. Hence we use the process η from (4) but transform the marginal distributions to the generalized Pareto distribution $1 - 1/x$, $x \geq 1$:

$$\xi(s_1, s_2) := \frac{1}{1 - \exp\left\{-\frac{1}{\eta(s_1,s_2)}\right\}} \tag{5}$$

for (s_1, s_2) in the area.

The last step is a further transformation of the marginal distribution that adapts the process to the local shape (γ), scale (a) and shift (b) parameters. These parameters can be estimated from each station separately, using the local sample. However, the resulting estimates may not be accurate enough, due to the small sample size (there is a large number of days with no rain). To increase precision, it is often assumed in the hydrological and climatological literature that the shape parameter γ is constant over the region of interest (e.g., NERC [23]; Alila [1]; Gellens [19]; Fowler and Kilsby [18]). A reliable estimate of γ is then obtained using all extreme values (usually in the literature this concerns the seasonal or annual maxima) in the region.

Here we use the average of the local estimates of γ. We found the value $\hat{\gamma} = 0.1082$. This value is comparable with the estimates of the shape parameter found for daily maximum rainfall in the winter half-year (October-March) in the Netherlands (Buishand [7]) and Belgium (Gellens [19]). Of course our model allows γ to vary over the area.

The final transformation results into the process

$$X(s_1, s_2) := \hat{a}_{(s_1,s_2)}(n/k)\left(\frac{\xi(s_1, s_2)^{\hat{\gamma}_{n,k}} - 1}{\hat{\gamma}_{n,k}}\right) + \hat{b}_{(s_1,s_2)}(n/k). \tag{6}$$

Remark 3.1. *The estimation for γ, a and b (c.f. Proposition 2.1)at any location is based on the "extreme" part of the local sample, i.e. the upper k order statistics of that sample.*

In the asymptotic theory, when the sample size n *is going to infinity,* k *will go to infinity:* $k = k(n) \to \infty$*; but of lower order than* n*:* $k(n)/n \to 0, n \to \infty$.

The estimation of the shift $b_{(s_1,s_2)}(n/k)$ *is particularly simple:* $\hat{b}_{(s_1,s_2)}(n/k)$ *is the* k*-th largest order statistics of the local sample.*

There are various estimators of γ *and* $a_{(s_1,s_2)}(n/k)$ *that converge at speed* $k^{-1/2}$*. In the present application we use the so-called moment estimator for* γ *(c.f. e.g. de Haan and Ferreira [13], section 3.9) and the accompanying estimator for* $a_{(s_1,s_2)}(n/k)$ *(c.f. e.g. de Haan and Ferreira [13], section 4.2).*

As explained before, to obtain a global shape parameter, we take the average of the local estimates of γ *among all the stations. However, we keep the local estimates of the scale and shift at each station.*

The number k *of upper order statistics used for the estimation of the shape parameter* γ *(*$k = 125$*, see Figure 2), is used also for estimating the scale* a *and the shift* b *throughout the area. The sample size* n *is 2730.*

The process (6) provided the simulated (extreme) rainfall in the area.

4 Simulating a Day of Rainfall

On an arbitrary day, there will be "extreme" rainfall in part of the area and "non-extreme" rainfall (or no rainfall at all) in the rest of the area.

We achieve this in the simulation as follows: on the one hand, we simulate the process (6) for the whole area; on the other hand, we choose at random a day out of the 30*(30+31+30)=2730 days of observed rainfall

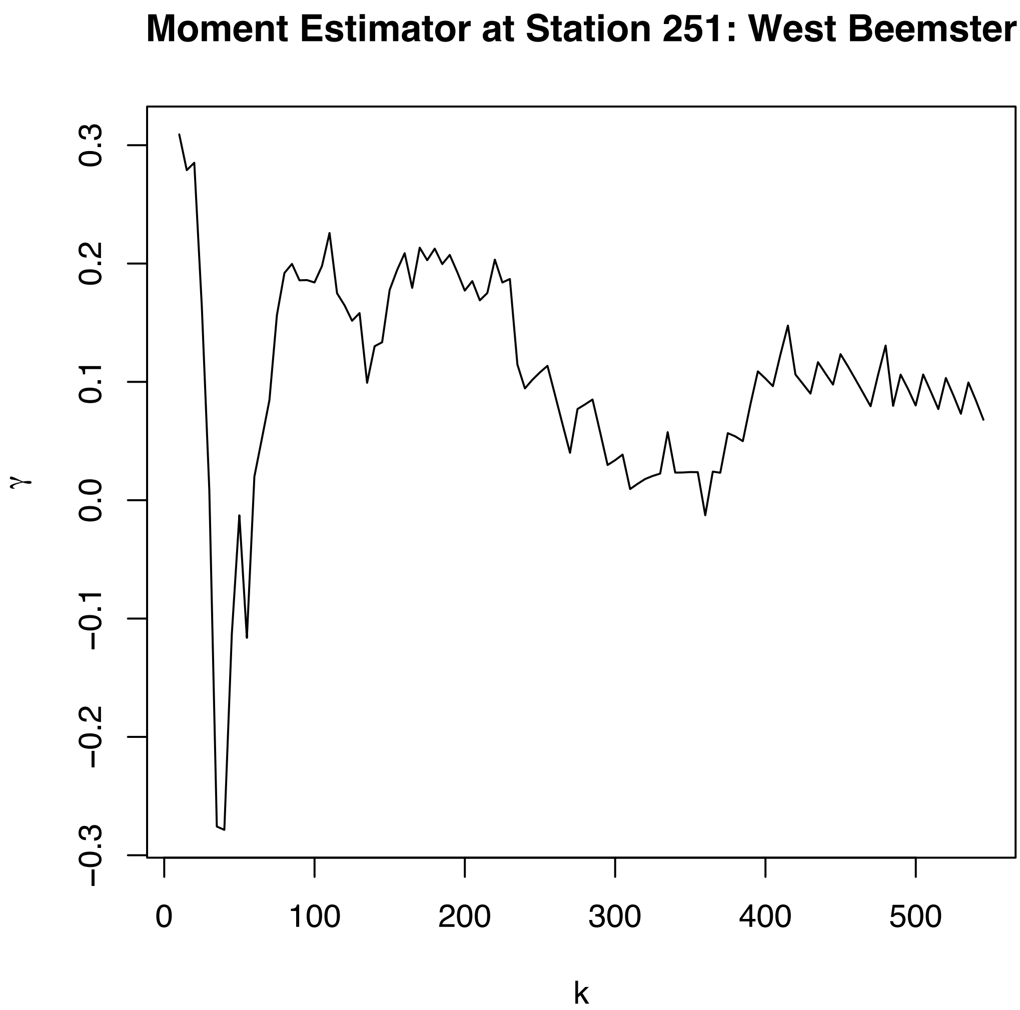

Figure 2: The Moment Estimator of γ at Station 251: West Beemster

and we connect the two as follows:

For each station we check whether the observed rainfall on the chosen day is larger than the shift parameter $\hat{b}_{(s_1,s_2}(n/k)$ for that station. If so, we use (6) (i.e., the simulated process) to get the rainfall at that station. If not, we just use the observed rainfall for the chosen day at that station.

How do we extend this to obtain the rainfall in the entire area?

First we connect the monitoring stations with each other, so as to cover the area with Triangels, see Figure 3 (The station names corresponding to the numbers are given in Table 1). We write Triangles since later on we shall also deal with smaller triangles, also we write Vertex and Edge for a vertex and edge of a Triangle. Any Triangle can be extreme or non-extreme.

1. Non-extreme: this is the case if all Vertices of the Triangle are non-extreme. The rainfall in such a Triangle is just a linear function whose value at the Vertices are the observed values.

2. Extreme: all other cases. In that case the rainfall is mainly determined by the process (6) where the functions $a_{(s_1,s_2)}(n,k)$ and $b_{(s_1,s_2)}(n,k)$ on the Triangle are chosen as linear functions whose value at the Vertices are the values obtained by local estimation.

More specifically we proceed as follows:

2.a) Subdivide each Edge into d intervals of equal length. Connect the separating points on the Edges with each other using lines parallel to the Edges as in Figure 4.

This results into d^2 triangles inside a Triangle. We used $d = 5$ in the simulation.

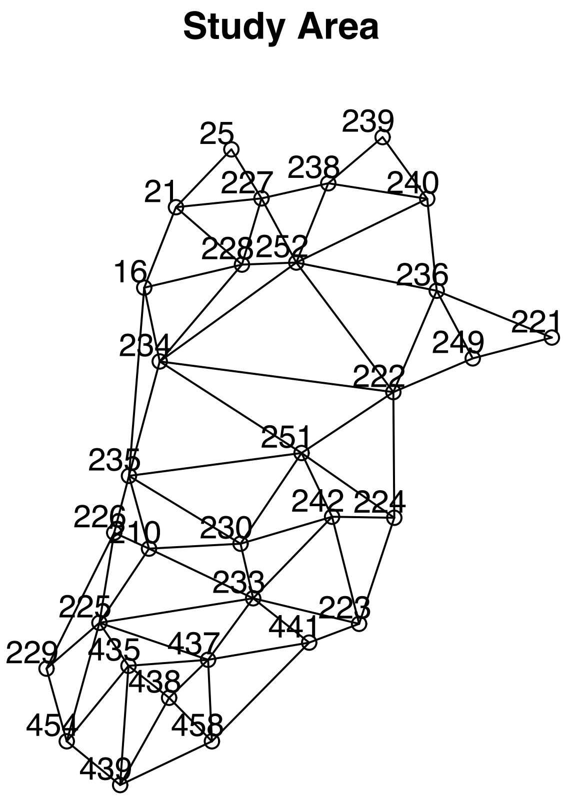

Figure 3: The Triangles connecting the observation stations

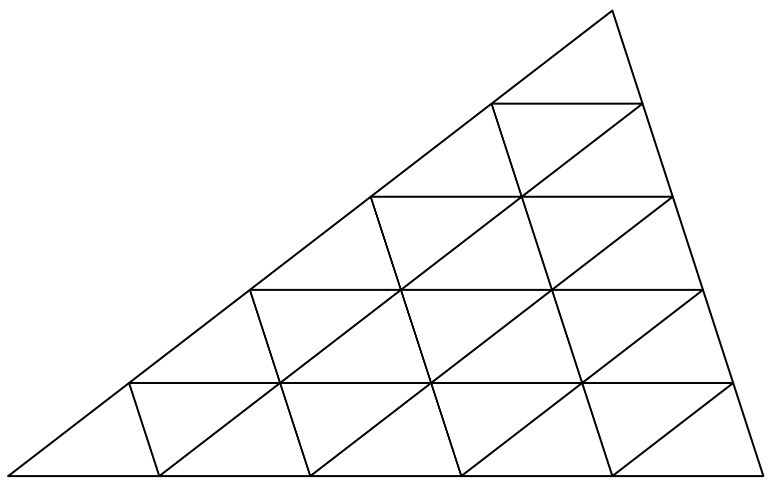

Figure 4: Division of a Triangle into d^2 small triangles ($d = 5$)

2.b) Next we determine the rainfall process in each vertex (i.e. vertex of a triangle). For Vertices we already determined the process. For the vertices, there are two cases.

2.b.1 On an Edge connecting two non-extreme Vertices in an extreme Triangle, the rainfall is chosen to be the linear function whose values at the Vertices are the observed values. This determines the rainfall for all vertices on such an Edge. The process (6) plays no role.

2.b.2 The rainfall for every other vertex in an extreme Triangle is determined by the process (6).

2.c) In order to carry out the numerical integration we simplify the rainfall process on each triangle in an extreme Triangle. The rainfall in each triangle is given as a linear function whose value at the vertices is the one obtained in part **2.b**.

This is the way we obtained a day of rainfall. Note that the process is continuous and that it is easy to integrate numerically.

We remark that on 2299 out of the 2730 days of observation, none of the Vertices (stations) is extreme, so that no simulation is necessary. On the other hand, there are 44 days on which all Triangles are extreme, so that the whole area is simulated.

5 Estimation of the Dependence Parameter

One problem remains: we do not know β, the global dependence parameter in (2). It has to be estimated. This can be done along the lines indicated in de Haan and Pereira [15].

We need to calculate the two-dimensional marginal distributions of the process η (defined in (2))at locations (u_1, u_2) and (v_1, v_2), say. This is done in Appendix B. The result is as follows: for x, y real with $h := |u_1 - v_1| + |u_2 - v_2|$,

$$\begin{aligned} &P(\eta(u_1, u_2) \le e^x, \eta(v_1, v_2) \le e^y) \\ =& \exp\left\{ - \left(e^{-x}\Phi\left(\frac{\sqrt{\beta h}}{2} + \frac{y-x}{\sqrt{\beta h}} \right) + e^{-y}\Phi\left(\frac{\sqrt{\beta h}}{2} + \frac{x-y}{\sqrt{\beta h}} \right) \right) \right\}, \end{aligned} \tag{7}$$

where Φ is the standard normal distribution function. Taking $x = y = 0$, we find

$$P(\eta(u_1, u_2) \le 1, \eta(v_1, v_2) \le 1) = \exp\left\{ -2\Phi\left(\frac{\sqrt{\beta h}}{2} \right) \right\},$$

and consequently

$$\beta = \frac{4}{h}\left(\Phi^{\leftarrow}\left(-\frac{1}{2}\log P(\eta(u_1, u_2) \le 1, \eta(v_1, v_2) \le 1) \right) \right)^2.$$

Hence we can estimate β if we know how to estimate

$$L_{(u_1,u_2),(v_1,v_2)}(1,1) := -\log P(\eta(u_1, u_2) \le 1, \eta(v_1, v_2) \le 1).$$

This is a problem of two-dimensional extreme value theory that has been solved by Huang and Mason (cf. Huang [21], Drees and Huang [16]).

Let the continuous process X be in $\mathcal{D}$ (c.f. beginning of Section 2.3). Let $X_1, X_2, \cdots$ be i.i.d. copies of X. Write $\{X_{i,n}(s_1, s_2)\}_{i=1}^n$ for the order statistics at location (s_1, s_2). Then the estimator

$$\hat{L}^{(k)}_{(u_1,u_2),(v_1,v_2)}(1,1) := \frac{1}{k}\sum_{j=1}^{n} 1_{\left\{X_j(u_1,u_2) \ge X_{n-k+1,n}(u_1,u_2) \quad \text{or} \quad X_j(v_1,v_2) \ge X_{n-k+1,n}(v_1,v_2)\right\}}$$

is consistent provided $k = k(n) \to \infty$, $k(n)/n \to 0$, $n \to \infty$. It is asymptotically normal under certain mild extra conditions.

Now indicate the monitoring stations by the numbers $1, 2, \cdots, N(N = 32)$ and define for $p < q \leq N$,

$$\hat{\beta}_{p,q} = \frac{4}{h}\left(\Phi^{\leftarrow}\left(\frac{1}{2}\hat{L}^{(k(p,q))}_{(u_1,u_2),(v_1,v_2)}(1,1)\right)\right)^2,$$

where (u_1, u_2) and (v_1, v_2) are the coordinates of station p and q respectively, $k(p, q)$ is the number of higher order statistics used in the estimation. Our estimator for β is

$$\hat{\beta} := \frac{2}{N(N-1)}\sum_{q=2}^{N}\sum_{p=1}^{q-1}\hat{\beta}_{p,q}$$

(consistent and asymptotically normal).

We found that $\hat{\beta} = 0.04277$.

Remark 5.1. *Note that the estimators $\hat{\gamma}$, $\hat{a}$ and $\hat{b}$ come from one-dimensional extreme value theory, the estimator $\hat{\beta}$ comes from finite-dimensional extreme value theory and the process η comes from extreme value theory in $C[0, 1]$.*

6 Result

Our purpose is to study extremes of the total rainfall in North Holland. In particular we want to determine how severe the areal rainfall is that occurs once in 100 years. To be precise, it is once in 100*(30+31+30)=9100 days. In other words, we are studying the 1-1/9100 quantile of the daily total rainfall in the area. This quantile will be briefly indicated as the 100-year quantile.

Before presenting the simulation result, we would like to introduce some statistics and results for separate stations.

Take Station 251 - West Beemster - as an example (it is located in the middle of the area, and considered as the origin point when simulating the dependence process). The largest observed rainfall in the 30 years is 68.2 mm.

By fitting the GPD with shape parameter $\hat{\gamma} = 0.1082$ to the observed extreme daily rainfall amounts at West Beemster, we can estimate the 1-1/9100 quantile for this station. The point estimator is 63.0 mm.

It can also be done for the other stations to find the 1-1/9100 quantile in each monitoring station. The results are given in Table 1.

From the table, we can get that the average 1-1/9100 quantile among all the stations is 66.9 mm.

The simulation procedure in Section 4 has been repeated 91,000 times. This results in a sample of 91,000 days rainfall in North Holland. For each day we calculate the total rainfall as the numerical integral of the rainfall process on the area. We take the 10th largest order statistic of this sample, i.e. we determine the 1-1/9100 sample quantile of the integrated rainfall. Dividing by the total area, 2010 km^2, we get the average rainfall in the area. We replicate this procedure 10 times. The 10 simulated quantiles are given in Table 2.

The sample mean of the simulated quantiles is 59.5 mm, with sample standard deviation 3.63 mm. Hence the standard deviation of the sample mean is 1.15 mm.

The quantile for the area-average rainfall is thus smaller than the average of the corresponding quantile for the individual measuring stations. The areal reduction factor equals $ARF = 0.87$. It is remarkable that

Table 1: Estimation of the 100-year Quantile for Each Station

Station No.	Station Name	100-year Quantile (mm)
16	PETTEN	64.7
21	CALLANTSOOG	75.8
25	DE KOOY	74.0
210	BEVERWIJK	66.5
221	ENKHUIZEN	54.1
222	HOORN	54.0
223	SCHELLINGWOUDE	69.6
224	EDAM	59.9
225	OVERVEEN	67.6
226	WIJK AAN ZEE	67.1
227	ANNA PAULOWNA	78.0
228	SCHAGEN	71.2
229	ZANDVOORT	63.7
230	ZAANDIJK	73.5
233	ZAANDAM (HEMBRUG)	65.8
234	BERGEN	78.3
235	CASTRICUM	67.8
236	MEDEMBLIK	64.1
238	DE HAUKES	69.0
239	DEN OEVER	74.6
240	KREILEROORD	65.0
242	PURMEREND	73.0
249	HOOGKARSPEL	52.4
251	WEST BEEMSTER	63.0
252	KOLHORN	71.3
435	HEEMSTEDE	60.2
437	LIJNDEN	69.5
438	HOOFDDORP	65.5
439	ROELOFARENDSVEEN	58.6
441	AMSTERDAM	67.8
454	LISSE	72.2
458	AALSMEER	64.7

Table 2: Simulated 100-Year Quantiles of Area-Average Rainfall:

Sample No.	1	2	3	4	5	6	7	8	9	10
100-Year Quantiles (mm)	58.8	57.0	56.2	61.6	56.8	65.5	65.0	60.7	58.9	54.8

from the graph in the UK Flood Studies Report (see NERC [23]), a similar value of ARF is found for an area of 2010 km^2. The latter refers to annual maximum rainfall rather than seasonal maximum rainfall.

7 Conclusion

The theory of extremes of continuous processes was used to estimate the 100-year quantile of the daily area-average rainfall over North Holland. The estimation of this quantile was done by simulating the daily process.

Regions with large rainfall were generated using a specific max-stable spatial process. It was argued that direct simulation from the excursion process is not feasible.

The estimated 100-year quantile for the areal average rainfall turns out to be 11% lower that the average 100-year quantile of the 32 measurement stations.

Acknowledgement We thank J. Nellestijn for producing Figure 1.

Appendix A

Proof of shift stationarity of η

Consider the Ornstein-Uhlenbeck process (Breiman [5], Chapter 16, § 1). A representation of that process convenient for our purposes is as follows: for $s \in \mathbb{R}$

$$Y(s) = 1_{s\geq 0} e^{-s} \left(N + \int_0^s e^{u/2} dB_+(u) \right) + 1_{s<0} e^{-|s|} \left(N + \int_0^{|s|} e^{u/2} dB_-(u) \right)$$

with N, B_+ and B_- independent; N is a standard normal random variable, B_+ and B_- are standard Brownian motions. It is easy to check that indeed $EY(s_1)Y(s_2) = e^{-|s_1-s_2|}$, for $s_1, s_2 \in \mathbb{R}$.

Now in a way very similar to Brown and Resnick [6], it can be proved that the process Y is in the maximum domain of attraction of the process

$$\bigvee_{i=1}^{\infty} Z_i \exp\left\{ W_i(s) - |s|/2 \right\} \tag{8}$$

with the point process $\{Z_i\}$ and the i.i.d. processes W_i as in (3). More precisely, with i.i.d. copies $Y_1, Y_2, \cdots$ from Y, the sequence of processes

$$\left\{ \bigvee_{i=1}^{n} b_n \left(Y_i \left(\frac{s}{b_n^2} \right) - b_n \right) \right\}_{s\in\mathbb{R}} \tag{9}$$

with $b_n = (2\log n - \log\log n - \log(4\pi))^{1/2}$ converges weakly to the process (8) in $C[0,1]$. (c.f. Einmahl and Lin [17])

Since for each n, the process (9) is stationary, the process (8) must be stationary as well.

It follows that the process η from (2) is shift stationary.

Appendix B

Proof of two-dimensional joint distribution of η (see (7)).

We need the following Lemma.

Lemma B.1. *Suppose N is normally distributed with mean 0, variance u, then with non-random constants $a > 0$ and b,*

$$Ee^{N-u/2}\Phi(aN+b) = \Phi\left(\frac{au+b}{\sqrt{a^2u+1}}\right). \tag{10}$$

Proof Suppose N_1 is standard normally distributed, and independent of N, then we have

$$Ee^{N-u/2}1_{N_1\leq aN+b} = E_N E(e^{N-u/2}1_{N_1\leq aN+b}|N) = Ee^{N-u/2}\Phi(aN+b),$$

which is the left side of (10). By Fubini's Theorem, it can be recalculated in the following way

$$\begin{aligned}
&Ee^{N-u/2}1_{N_1\leq aN+b}\\
=&E_{N_1}E(e^{N-u/2}1_{N_1\leq aN+b}|N_1)\\
=&E_{N_1}\int_{\frac{N_1-b}{a}}^{\infty} e^{t-u/2}\frac{1}{\sqrt{2\pi u}}e^{-\frac{t^2}{2u}}\,dt\\
=&E_{N_1}\int_{\frac{N_1-b}{a}}^{\infty} \frac{1}{\sqrt{2\pi u}}e^{-\frac{(t-u)^2}{2u}}\,dt\\
=&E_{N_1}\left(1-\Phi\left(\frac{N_1-b}{a\sqrt{u}}-\sqrt{u}\right)\right).
\end{aligned}$$

By a similar trick - introducing a standard normal variable N_2 independent of N_1, the calculation can be finished to prove the lemma.

$$
\begin{aligned}
&E_{N_1}\left(1-\Phi\left(\frac{N_1-b}{a\sqrt{u}}-\sqrt{u}\right)\right)\\
=&E_{N_1}E(1_{N_2\geq \frac{N_1-b}{a\sqrt{u}}-\sqrt{u}}|N_1)\\
=&E_{N_1,N_2}1_{N_2\geq \frac{N_1-b}{a\sqrt{u}}-\sqrt{u}}\\
=&P(N_2\geq \frac{N_1-b}{a\sqrt{u}}-\sqrt{u})\\
=&\Phi\left(\frac{au+b}{\sqrt{a^2u+1}}\right)
\end{aligned}
$$

□

In de Haan and Ferreira [13], section 9.8, the two-dimensional joint distribution has been calculated for the stochastic process with one-dimensional index. We state it as the following proposition.

Proposition B.1. *Suppose* $\{\tilde{\eta}(s)\}_{s\in\mathbb{R}}$ *is defined as in (8). Then for* $x, y \in \mathbb{R}$ *and* $s_1, s_2 \in \mathbb{R}$*,*

$$
\begin{aligned}
&-\log P(\tilde{\eta}(s_1)\leq e^x, \tilde{\eta}(s_2)\leq e^y)\\
=&e^{-x}\Phi\left(\frac{\sqrt{|s_1-s_2|}}{2}+\frac{-x+y}{\sqrt{|s_1-s_2|}}\right)+e^{-y}\Phi\left(\frac{\sqrt{|s_1-s_2|}}{2}+\frac{x-y}{\sqrt{|s_1-s_2|}}\right).
\end{aligned}
$$

By applying this, we have as in the proof of Proposition B.1 (c.f. de Haan and Ferreira [13], Section 9.8),

$$
\begin{aligned}
&-\log P(\eta(u_1,u_2)\leq e^x, \eta(v_1,v_2)\leq e^y)\\
=&E\max\left(e^{W_1(\beta u_1)+W_2(\beta u_2)-(|\beta u_1|+|\beta u_2|)/2-x}, e^{W_1(\beta v_1)+W_2(\beta v_2)-(|\beta v_1|+|\beta v_2|)/2-y}\right)\\
=&E_{W_1}E\left(\max\left(e^{W_1(\beta u_1)+W_2(\beta u_2)-(\beta|u_1|+\beta|u_2|)/2-x}, e^{W_1(\beta v_1)+W_2(\beta v_2)-(\beta|v_1|+\beta|v_2|)/2-y}\right)|W_1\right)\\
=&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
&+Ee^{-y+W_1(\beta v_1)-\beta|v_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{x-y+W_1(\beta v_1)-W_1(\beta u_1)-\beta|v_1|/2+\beta|u_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)
\end{aligned}
\tag{11}
$$

Now we can calculate the two parts in (11) separately. Without losing generality, we only focus on the first part.

Case 1: $0 \le u_1 \le v_1$

In this case $e^{-x+W_1(\beta u_1)-\beta|u_1|/2}$ is independent of the other part. Hence

$$
\begin{aligned}
&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}E\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x-(W_1(\beta v_1)-W_1(\beta u_1)-\beta(v_1-u_1)/2)}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}P\left(N\le\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x-(W_1(\beta v_1)-W_1(\beta u_1)-\beta(v_1-u_1)/2)}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|+\beta(v_1-u_1)}}{2}+\frac{y-x}{\sqrt{\beta|u_2-v_2|+\beta(v_1-u_1)}}\right)
\end{aligned}
$$

Case 2: $0 \le v_1 < u_1$

Note that $Ee^{W_1(\beta v_1)-\beta v_1/2}=1$ and $W_1(\beta v_1)$ is independent of $W_1(\beta u_1)-W_1(\beta v_1)$, we have

$$
\begin{aligned}
&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}Ee^{W_1(\beta u_1)-W_1(\beta v_1)-\beta(u_1-v_1)/2}\\
&\cdot\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)
\end{aligned}
$$

Since $W_1(\beta u_1)-W_1(\beta v_1)$ is normally distributed with mean 0, variance $\beta(u_1-v_1)$, we can apply Lemma B.1 with the constants $a=1/\sqrt{\beta|u_2-v_2|}$, $u=\beta(u_1-v_1)$ and

$$
b=\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x-\beta u_1/2+\beta v_1/2}{\sqrt{\beta|u_2-v_2|}}.
$$

The final result is

$$
\begin{aligned}
&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|+\beta(u_1-v_1)}}{2}+\frac{y-x}{\sqrt{\beta|u_2-v_2|+\beta(u_1-v_1)}}\right).
\end{aligned}
$$

Case 3: $v_1 < u_1 < 0$ and $u_1 \leq v_1 < 0$

These two cases are similar to Case 1 and 2 respectively. The final results are all the same as following

$$\begin{aligned}
&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|+\beta|u_1-v_1|}}{2}+\frac{y-x}{\sqrt{\beta|u_2-v_2|+\beta|u_1-v_1|}}\right).
\end{aligned}$$

Case 4: u_1 and v_1 have different signs.

In this case $W_1(\beta u_1)$ and $W_1(\beta v_1)$ are independent, we can calculate the expectation with respect to $W_1(\beta v_1)$ first, then with respect to $W_1(\beta u_1)$.

$$\begin{aligned}
&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}Ee^{W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|+\beta|v_1|}}{2}+\frac{y-x+W_1(\beta u_1)-\beta|u_1|/2}{\sqrt{\beta|u_2-v_2|+\beta|v_1|}}\right)
\end{aligned}$$

Now we can again apply Lemma B.1 with the constants $a = 1/\sqrt{\beta|u_2-v_2|+\beta|v_1|}$, $u = \beta|u_1|$ and

$$b = \frac{\sqrt{\beta|u_2-v_2|+\beta|v_1|}}{2}+\frac{y-x-\beta|u_1|/2}{\sqrt{\beta|u_2-v_2|+\beta|v_1|}}.$$

to get

$$\begin{aligned}
&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|+\beta(|u_1|+|v_1|)}}{2}+\frac{y-x}{\sqrt{\beta|u_2-v_2|+\beta(|u_1|+|v_1|)}}\right)
\end{aligned}$$

Notice that due to the different signs of u_1 and v_1, $|u_1-v_1| = |u_1|+|v_1|$.

By defining $h = |u_1-v_1|+|u_2-v_2|$, all these cases can be combined as

$$\begin{aligned}
&Ee^{-x+W_1(\beta u_1)-\beta|u_1|/2}\Phi\left(\frac{\sqrt{\beta|u_2-v_2|}}{2}+\frac{y-x+W_1(\beta u_1)-W_1(\beta v_1)-\beta|u_1|/2+\beta|v_1|/2}{\sqrt{\beta|u_2-v_2|}}\right)\\
=&e^{-x}\Phi\left(\frac{\sqrt{\beta h}}{2}+\frac{y-x}{\sqrt{\beta h}}\right)
\end{aligned}$$

Symmetrically, the second part of (11) can be simplified as

$$e^{-y}\Phi\left(\frac{\sqrt{\beta h}}{2}+\frac{x-y}{\sqrt{\beta h}}\right).$$

Combining these two parts, we proved (7).

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